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SHORTER COMMUNICATIONS

AN EXAMINATION OF THE METHOD OF REGIONAL AVERAGING FOR RADIATIVE TRANSFER BETWEEN CONCENTRIC SPHERES*

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NOMENCLATURE

B,	frequency-integrated Planck function;
I,	frequency-integrated specific intensity;
I_0,	$\int_{4\pi} I \, d\omega$;
I_{2rr},	$\int_{4\pi} l_r l_r I \, d\omega$;
J_r,	$I_{2rr} - I_0/3$;
K,	absorption coefficient;
l_i,	direction cosine;
$P_1(\tau_2, \tau_1)$,	see equation (18);
$P_2(\tau_2, \tau_1)$,	see equation (24);
q_r,	radial component of radiative flux vector;
$Q(\tau_2, \tau_1)$,	see equation (14);
r,	radial distance;
$R_1(\tau, \tau_1)$,	see equation (15);
$R_2(\tau, \tau_1)$,	see equation (19);
$R_3(\tau, \tau_1)$,	see equation (20);
S,	heat generated per unit volume;
$\bar{S}$,	S/K;
λ,	$[1 - (r_1/r)^2]^{\frac{1}{2}}$;
τ,	Kr;
ϕ_H,	$\pi(B - B_W)/\bar{S}\tau_2$;
ϕ_W,	$(B_{W1} - B)/(B_{W1} - B_{W2})$;
ω,	solid angle.

Subscripts

W,	value on a boundary;
1, 2,	inner, outer spheres.

INTRODUCTION

THE EXACT solution of radiative transfer problems is prohibitively difficult for all except a limited number of simple cases. Approximate transfer equations (which can be derived in a number of alternative ways) have therefore been widely used [1]. In planar geometries, the use of these equations has been very successful. However, in nonplanar geometries, the approximations can break down. This problem has been described in a number of recent papers [2–4].

A recent approach to the problem is the method of regional averaging due to Chou and Tien [5, 6] who have adapted a method developed by Lees [7] for kinetic theory. In this paper, the accuracy of the method of regional averaging is examined for the case of spherical symmetry. Certain curvature terms which were neglected in the original papers on this method [5, 6] are here included in the analysis. In addition, several further cases are investigated.

THE APPROXIMATE EQUATIONS FOR SPHERICAL GEOMETRY

The basic equation for the specific intensity of radiation for a medium in local thermodynamic equilibrium is

$$\frac{\partial I}{\partial x_i} l_i = K(B - I) \tag{1}$$

where the l_i are the direction cosines of the ray referred to a cartesian system of axes, x_i, B is the frequency-integrated Planck function and K is the absorption coefficient of the medium which is here assumed independent of frequency for simplicity.

The formulation of the method of regional averaging has been given by Chou [5, 6]. It consists of multiplying equation (1) in turn by each of the various products of the direction cosines and integrating over solid angle to obtain a well known [9], infinite set of moment equations. These equations are then expressed in the appropriate coordinate system and truncated by means of a set of approximate relations. The approximate relations are obtained as follows. The solid angle about any point in the medium is divided up into regions according to the geometry of the problem (see Fig. 1). The approximation is then made that, within

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each region, the specific intensity is independent of angle. Upon introducing this assumption into the definitions of the various moments of the specific intensity, each can be expressed in terms of the regionally constant value of specific intensity. Elimination of these values then gives the desired set of relations connecting the moments.

Figure 1 shows the geometry of the problem under consideration. The method of regional averaging will be applied for each of three different approximations for the distribution of intensity over solid angle, namely 3-region, 2-region and Milne–Eddington. Details of the derivation of the approximate equations in each of these cases has been given in [8].

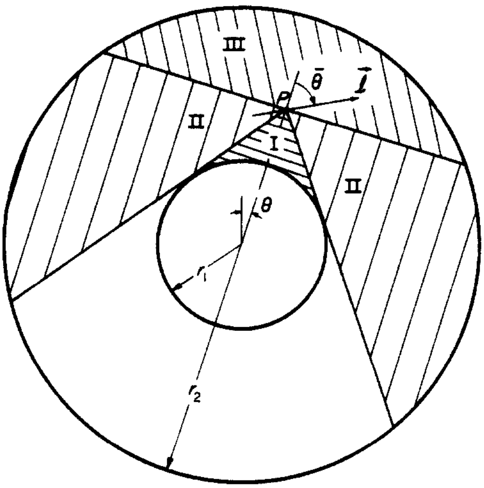

FIG. 1. Schematic drawing of the geometry used.

3-Region. In the 3-region approximation, originally proposed by Chou [5], the specific intensity is taken as constant within each of regions I, II and III. The approximate equations are found to be

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 q_r) = K(4\pi B - I_0) \tag{2}$$

$$\frac{\mathrm{d}I_{2rr}}{\mathrm{d}r} + \frac{3}{r}J_r = -Kq_r \tag{3}$$

$$\frac{3}{4}\frac{\mathrm{d}}{\mathrm{d}r}(\lambda J_r) + 3\frac{\lambda}{r}J_r + \frac{1}{2}\frac{\mathrm{d}q_r}{\mathrm{d}r} = -KJ_r + \frac{K}{3}(4\pi B - I_0) \tag{4}$$

where

$$J_r \equiv I_{2rr} - I_0/3, \qquad I_0 \equiv \int_{4\pi} I\,\mathrm{d}\omega, \qquad q_r \equiv \int_{4\pi} l_r I\,\mathrm{d}\omega,$$

$$I_{2rr} \equiv \int_{4\pi} l_r l_r I\,\mathrm{d}\omega \quad \text{and} \quad \lambda \equiv [1 - (r_1/r)^2]^{\frac{1}{2}}.$$

The equations derived by Chou and Tien [5, 6] differ from those above in the neglect of the quantities $(3/r)J_r$ and $3(\lambda/r)J_r$ which appear in equations (3) and (4) respectively.

The appropriate boundary conditions for gray, emitting and reflecting surfaces are

$$\text{at } r = r_1, \qquad \varepsilon_1 I_0 + 2(2 - \varepsilon_1)q_r = 4\pi\varepsilon_1 B_{W1}, \qquad J_r = 0 \tag{5}$$

and

$$\text{at } r = r_2, \qquad \varepsilon_2 I_0 - 2(2 - \varepsilon_2)q_r = 4\pi\varepsilon_2 B_{W2} - 3J_r\frac{(2 - \varepsilon_2)}{1 + \lambda_2} \tag{6}$$

where $\lambda_2 \equiv [1 - (r_1/r_2)^2]^{\frac{1}{2}}$, ε is the wall emissivity and B_W is the black-body emissive power of the wall.

2-Region. It may be seen from Fig. 1 that whereas the intensity will, in most cases, change discontinuously between regions I and II, there is no discontinuity between regions II and III. This suggests that combining regions II and III may be a satisfactory approximation. However, this approximation requires that, at the outer boundary, the intensity in region II must be equal to the intensity leaving the outer sphere. In general, this is not physically correct. There are, however, cases where this outer boundary condition is satisfied, namely (a) the planar limit $r_1/r_2 \to 1$; (b) an optically thin medium in the absence of heat generation; (c) an optically thick medium; (d) an outer sphere of unbounded radius in the absence of heat generation (in this case, except in a region close to the inner sphere, the gas will take up the temperature of the outer sphere, with the result that the intensity near the outer sphere is isotropic). Case (d) clearly has relevance to the problem of the upstream radiation from a curved shock layer.

The differential equations and their boundary conditions are found to be

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 q_r) = K(4\pi B - I_0) \tag{7}$$

$$\frac{\mathrm{d}I_0}{\mathrm{d}r} + 2\lambda\frac{\mathrm{d}q_r}{\mathrm{d}r} + \frac{2q_r}{r}\left(\frac{1}{\lambda} + 2\lambda\right) = -3Kq_r, \tag{8}$$

$$\text{at } r = r_1, \qquad \varepsilon_1 I_0 + 2(2 - \varepsilon_1)q_r = 4\pi\varepsilon_1 B_{W1} \tag{9}$$

and

$$\text{at } r = r_2, \qquad \varepsilon_2 I_0 - 2q_r\frac{(2 - \varepsilon_2)}{1 + \lambda_2} = 4\pi\varepsilon_2 B_{W2}. \tag{10}$$

Milne–Eddington. The Milne–Eddington equations for a spherical geometry have been studied previously [2, 3, 6] and the solutions are discussed in this paper only for completeness. Although normally derived by other means [1, 9], the Milne–Eddington equations can be obtained from the formulation of the method of regional averaging by combining regions I and II (as pointed out by Chou [5, 6]). The equations are too well known to need presenting here (see [1, 2, 6 or 8]).

SOLUTION OF THE APPROXIMATE EQUATIONS

In order to compare with existing numerical solutions [10, 11] of the exact equations of radiative transfer, the approximate equations presented earlier were solved for two classes of spherically symmetric problems (a) black walls at different temperatures, no internal heat generation and K = constant; (b) black walls at equal temperatures, uniform internal heat generation and K = constant.

The equation for the overall energy balance is therefore

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(q_r r^2) = S \tag{11}$$

where S is the (uniform) heat source per unit volume. In case (a) it is zero. The solutions to the approximate equations together with equation (11) are as follows [8].

3-*Region*. For no heat generation ($S = 0$) one gets

$$\frac{q_r}{4\pi(B_{W1} - B_{W2})} = \frac{1}{Q(\tau_2, \tau_1)}\left(\frac{\tau_1}{\tau}\right)^2 \tag{12}$$

$$\phi_W \equiv \frac{B_{W1} - B}{B_{W1} - B_{W2}} = \frac{1}{Q(\tau_2, \tau_1)}\left\{2 + \frac{3}{\tau}(1 - \lambda^2) + \frac{9}{4}\frac{(1 - \lambda^2)}{\lambda\tau^2}[\exp(-\tfrac{4}{3}\lambda\tau) - 1] + 3\tau_1\left(1 - \frac{\tau_1}{\tau}\right) + R_1(\tau, \tau_1)\right\} \tag{13}$$

where,

$$Q(\tau_2, \tau_1) \equiv 2(2 - \lambda_2^2) + \frac{3\lambda_2}{\tau_2}(1 - \lambda_2) + \frac{9}{4}\frac{1 - \lambda_2}{\tau_2^2}[\exp(-\tfrac{4}{3}\lambda_2\tau_2) - 1] + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right) + R_1(\tau_1, \tau_2) \tag{14}$$

and

$$R_1(\tau, \tau_1) \equiv \int_{\tau_1}^{\tau}\left\{\frac{27}{4}\frac{\tau_1^2}{\lambda t^5}[\exp(-\tfrac{4}{3}\lambda t) - 1] + \frac{9\tau_1^2}{t^4}\right\}\mathrm{d}t. \tag{15}$$

In equations (12–15), $\tau \equiv Kr$ and the quantity λ in the integrand of equation (15) is based on t, i.e. it is $[1 - (\tau_1/t)^2]^{\frac{1}{2}}$. R_1 can clearly be written as the sum of a quadrature and certain expressions obtained by closed-form integration. However, it was found more accurate to perform the numerical integration on the above integrand.

For the case of heat generation with $B_{W2} = B_{W1} = B_W$, the corresponding expressions are

$$\frac{q_r}{S\tau_2} = \frac{1}{3}\frac{\tau}{\tau_2} - \frac{P_1(\tau_2, \tau_1)}{Q(\tau_2, \tau_1)}\left(\frac{\tau_1}{\tau}\right)^2. \tag{16}$$

$$\phi_H \equiv \frac{\pi(B - B_W)}{S\tau_2} = \frac{1}{4}\left\{\phi_W P_1(\tau_2, \tau_1) + \frac{1}{\tau_2} - \frac{2}{3}\frac{\tau_1}{\tau_2} - 2\frac{\tau_1}{\tau_2}R_3(\tau, \tau_1) - \frac{1}{2}\frac{\lambda^2\tau^2}{\tau_2} - \frac{2}{3}\frac{\tau_1}{\tau_2}R_2(\tau, \tau_1)\right\} \tag{17}$$

where

$$P_1(\tau_2, \tau_1) \equiv \frac{2}{3}\left(1 + \frac{\tau_1}{\tau_2}\right) + 2\frac{\tau_1}{\tau_2}R_3(\tau_2, \tau_1) + \frac{2}{3}\frac{\tau_1}{\tau_2}\frac{\lambda_2}{1 + \lambda_2}R_2(\tau_2, \tau_1) + \tfrac{1}{2}\tau_2\lambda_2^2, \tag{18}$$

$$R_2(\tau, \tau_1) \equiv \exp(-\tfrac{4}{3}\lambda\tau)/(\lambda\tau^4\tau_1)\int_{\tau_1}^{\tau} t^4\exp(\tfrac{4}{3}\lambda t)\,\mathrm{d}t, \tag{19}$$

$$R_3(\tau, \tau_1) \equiv \int_{\tau_1}^{\tau}\frac{1}{t}R_2(t, \tau_1)\,\mathrm{d}t, \tag{20}$$

$\bar{S} \equiv S/K$ and, as before, λ in an integrand is based on t.

2-*Region*. For the case of no heat generation, one has

$$\frac{q_r}{4\pi(B_{W1} - B_{W2})} = \frac{1}{4 + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right)}\left(\frac{\tau_1}{\tau}\right)^2,$$

$$\phi_W = \frac{2(1 + \lambda) + 3\tau_1\left(1 - \frac{\tau_1}{\tau}\right)}{4 + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right)} \tag{21}$$

The solutions for uniform heat generation and $B_{W2} = B_{W1} = B_W$ are

$$\frac{q_r}{S\tau_2} = \frac{1}{3}\frac{\tau}{\tau_2} - \frac{P_2(\tau_2, \tau_1)}{4 + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right)}\left(\frac{\tau_1}{\tau}\right)^2 \tag{22}$$

$$\phi_H = \frac{1}{4}\left\{\phi_W P_2(\tau_2, \tau_1) - \frac{\tau_1}{\tau_2}\left[\frac{2}{3} - \pi - \frac{\tau_1}{2} - \frac{1}{\tau_1} + \frac{8}{3}\lambda\frac{\tau}{\tau_1} + 2\arcsin\frac{\tau_1}{\tau} + \frac{1}{2}\frac{\tau^2}{\tau_1}\right]\right\} \tag{23}$$

where

$$P_2(\tau_2, \tau_1) \equiv \frac{\tau_1}{\tau_2}\left(\frac{2}{3} - \pi + 2\arcsin\frac{\tau_1}{\tau_2}\right) + \frac{2}{3}\frac{(2\lambda_2 + 1)^2}{(1 + \lambda_2)} + \tfrac{1}{2}\tau_2\lambda_2^2 \tag{24}$$

Milne–Eddington. The case of no heat generation gives

$$\frac{q_r}{4\pi(B_{W1} - B_{W2})} = \frac{1}{4 + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right) - 2\lambda_2^2}\left(\frac{\tau_1}{\tau}\right)^2,$$

$$\phi_W = \frac{2 + 3\tau_1\left(1 - \frac{\tau_1}{\tau}\right)}{4 + 3\tau_1\left(1 - \frac{\tau_1}{\tau_2}\right) - 2\lambda_2^2} \tag{25}$$

The results for uniform heat generation and $B_{W2} = B_{W1}$ have been given by Chou and Tien [6]. They are also given in [8].

DISCUSSION

The accuracy of the approximate solutions presented in the previous section is considered here. First, an anomaly which exists in the planar limit is described; second, a detailed comparison is made with existing exact solutions and, finally, a comment is made on the influence of certain curvature terms.

Planar limit

Approximate equations of the Milne–Eddington type are widely used for planar geometry. The evidence seems to be that they are accurate to within a few per cent for radiative flux [12, 13] but may be in error by as much as 20 per cent for emissive power [13, 8].

In the planar limit, region II disappears and the regional distributions of all the approximations become the same as the Milne–Eddington distribution (see Fig. 1). Consistent with this, the planar limits of the 2-region solutions are the same as those of the Milne–Eddington solutions. However, the planar limits of the 3-region solutions only correspond to the Milne–Eddington results in the case were $dq_r/d\tau = 0$ (i.e. no internal heat generation). Furthermore, the 3-region result for radiative heat flux, equation (16), gives different results depending on how the planar limit is taken. In particular, the limit $r_1/r_2 \to 1$ for finite r_2 leads to the prediction that $\frac{1}{4}$ of the total heat generated in the slab passes through wall 1, whereas taking the limit $r_1 \to \infty$ for fixed $r_2 - r_1$ yields the correct proportion, namely $\frac{1}{2}$.

The reason for this behaviour is that the truncation relation of the 3-region approximation does not reduce to the Milne–Eddington form since it relates higher order moments. This results in a singularity in the equations: the derivative in equation (4) becomes either zero or infinite and the boundary condition $J_r = 0$ can no longer be satisfied except in the case $I_0 = 4\pi B$ (i.e. no heat generation) when it is satisfied identically. It can be shown that a similar breakdown also occurs in Chou's results.

The case of no heat generation

Ryhming [10] has presented results for this case based on the exact equation of radiative transfer. In the first place, he gives asymptotic expressions for heat flux and temperature profile in the optically thick and optically thin limits. It can be shown that the 3-region and 2-region solutions approach the exact forms in these limits. The Milne–Eddington solutions are correct in the optically thick limit, but are incorrect (by up to a factor of 2) in the optically thin limit. It should also be noted that if one solves Chou's [5, 6] 3-region equations for this case, the results are asymptotically correct in the optically thick limit, but are incorrect in the optically thin limit (by up to a factor of $\frac{4}{3}$).

Comparisons between the exact and approximate results are made in Table 1 and Figs. 2 and 3. The 3-region results are in excellent agreement with the exact values, having a maximum error of 4 per cent (in q_r). The agreement in the case of the 2-region results is also seen to be good (within 10 per cent). In contrast, the Milne–Eddington results are in error by up to a factor of 2. For $\tau_2 = 100$, all the approximations are indistinguishable from the values plotted by Ryhming.

Table 1. Values of $q_r(\tau_1)/4\pi(B_{W1} - B_{W2})$ in the absence of heat generation

r_1/r_2	τ_2	Exact	3-Region	2-Region	Milne–Eddington
0·01	10·0	0·232	0·241	0·233	0·435
0·1		0·166	0·170	0·149	0·212
0·5		0·095	0·096	0·087	0·100
0·01	1·0	0·248	0·250	0·248	0·493
0·1		0·241	0·242	0·234	0·437
0·5		0·224	0·224	0·211	0·308

The case of uniform heat generation

Figures 4 and 5 compare the approximate solutions to the exact results of Sparrow, Usiskin and Hubbard [11] for $\tau_2 = 0{\cdot}1$. A similar comparison for $\tau_2 = 2{\cdot}0$ is given in [8]. It can be seen that, in this case, none of the approximations is uniformly satisfactory. The 3-region results are excellent

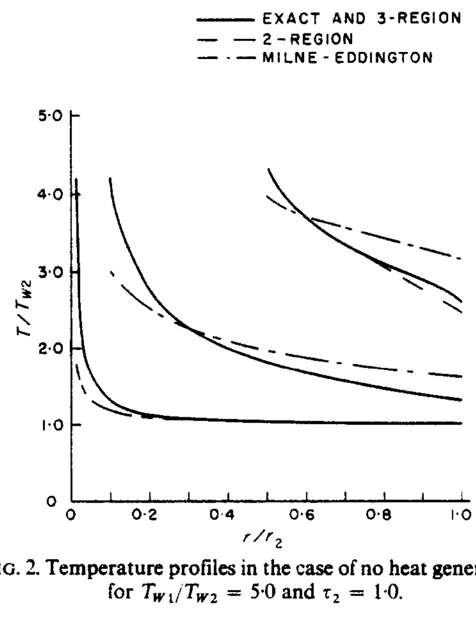

FIG. 2. Temperature profiles in the case of no heat generation for $T_{W1}/T_{W2} = 5{\cdot}0$ and $\tau_2 = 1{\cdot}0$.

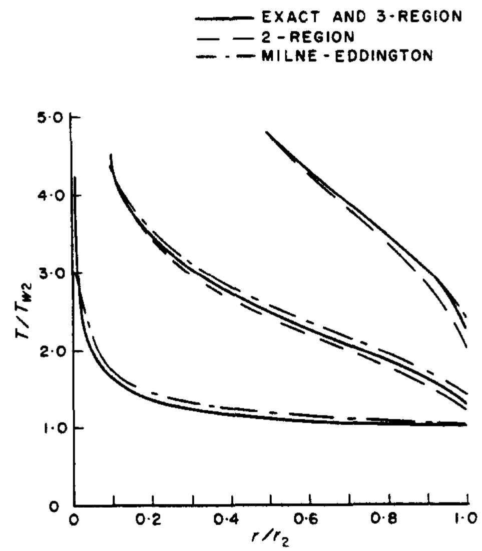

FIG. 3. Temperature profiles in the case of no heat generation for $T_{W1}/T_{W2} = 5{\cdot}0$ and $\tau = 10{\cdot}0$.

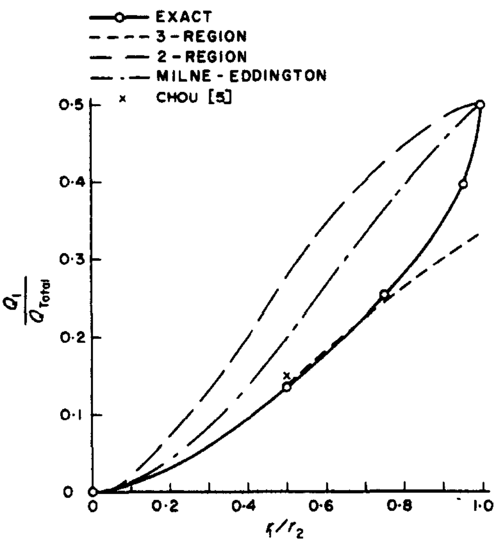

FIG. 4. Heat radiated through inner wall as a function of radius ratio in the case of uniform heat generation for $\tau_2 = 0{\cdot}1$.

up to a radius ratio of about 0·75, after which the planar breakdown sets in. The 2-region results are in error by as much as a factor of two due to a failure to represent the physical distribution of specific intensity at the outer boundary (as discussed earlier). The maximum error in the Milne–Eddington solution is 55 per cent. At $\tau_2 = 2{\cdot}0$ (see [8]) the approximate solutions are in better agreement with

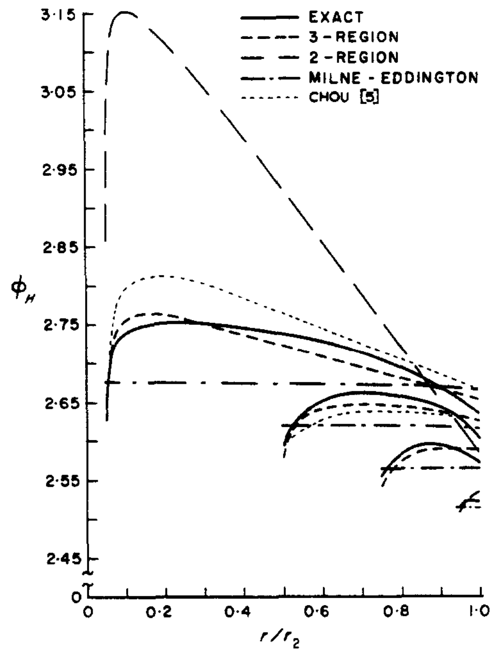

FIG. 5. Emissive power profiles in the case of uniform heat generation for $\tau_2 = 0{\cdot}1$.

the exact values; the 3-region results still exhibit planar breakdown, however.

Neglect of curvature terms in the 3-region approximation

Figures 4 and 5 include certain values calculated by Chou [5, 6] for the case of uniform heat generation. These values are in quite good agreement (12 per cent in q_r and 1 per cent in ϕ_H) with the exact values. For $\tau_2 = 2{\cdot}0$ and $r_1/r_2 = 0{\cdot}5$, the agreement in ϕ_H is still very good, but the error in heat flux is a little greater (15 per cent) (compare 3 per cent for the basic 3-region solution). Further, it is stated [5] that the solutions are poor when r_1/r_2 is small and τ_2 is large. The results just quoted were obtained from a 3-region approximation in which certain curvature terms were neglected and a "body-shape function" was replaced by a constant. The last approximation appears to improve the accuracy of the results since, in a case evaluated by Chou without this approximation ($\tau_2 = 0{\cdot}1$, $r_1/r_2 = 0{\cdot}5$), the error in wall heat flux rises from 12 per cent to 27 per cent (compare 4 per cent for the basic 3-region solution). The foregoing results suggest that the error involved in neglecting the curvature terms from equations (3) and (4) can be significant. In addition, it has been pointed out previously that, in the case of wall-driven radiation, errors of up to 33 per cent can occur when these terms are neglected and the gas is optically thin.

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LAMINAR PIPE FLOW IN A TRANSVERSE MAGNETIC FIELD WITH HEAT TRANSFER

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NOMENCLATURE

a,	radius of pipe;
A,	axial temperature gradient, $= \dfrac{2q}{\rho u_m C_p}$;
$A_n(\alpha)$,	function defined in equation (2);
B_0,	uniform magnetic field;
C_p,	specific heat;
d,	diameter of pipe, $= 2a$;
h,	heat-transfer coefficient;
$I_n(\alpha)$,	modified Bessel function of order n;
k,	thermal conductivity;
K,	non-dimensional pressure gradient, $= (\partial p/\partial x\, a^2)/u_m\,\mu$;
M,	Hartmann number, $= B_0 a(\sigma/\mu)^{\frac{1}{2}}$;
Nu,	Nusselt number, $= hd/k$;
p,	pressure;
Pe,	Péclet number, $= (\rho u_m a C_p)/k$;
q,	uniform wall heat flux;
r,	radial coordinate, non-dimensionalized by pipe radius;
T,	temperature;
T_m,	bulk mixing-cup temperature;
T_w,	wall temperature;
$u(r, \theta)$,	axial velocity, non-dimensionalized by u_m;

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